

Lambda-coalescents with dust component

Alexander Gnedin*, Alexander Iksanov† and Alexander Marynych‡

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Abstract

We consider the lambda-coalescent processes with positive frequency of singleton clusters. The class in focus covers, for instance, the beta(a, b)-coalescents with $a > 1$. We show that some large-sample properties of these processes can be derived by coupling the coalescent with an increasing Lévy process (subordinator), and by exploiting parallels with the theory of regenerative composition structures. In particular, we discuss the limit distributions of the absorption time and the number of collisions.

Keywords: absorption time, coupling, lambda-coalescent, number of collisions, regenerative composition structure, subordinator.

1 Introduction

The lambda-coalescent with values in partitions of n integers is a Markovian process $\Pi_n = (\Pi_n(t))_{t \geq 0}$ which starts at $t = 0$ with n singletons and evolves according to the rule: for each $t \geq 0$ when the number of clusters is m , each k tuple of them is merging in one cluster at probability rate

$$\lambda_{m,k} = \int_0^1 x^k (1-x)^{m-k} \nu(dx), \quad 2 \leq k \leq m, \quad (1)$$

where ν is a measure on the unit interval with finite second moment. The integral representation of rates (1) ensures that the processes Π_n can be defined consistently for all n , as restrictions of a coalescent process Π_∞ which starts with infinitely many clusters and assumes values in the set of partitions of \mathbb{N} , see [23]. The infinite coalescent Π_∞ may be regarded as a limiting form of Π_n as $n \rightarrow \infty$, and uniquely connected to a process with values in the infinite-dimensional space of partitions of a unit mass. The lambda-coalescents were introduced in the papers by Pitman [23] and Sagitov [25], where the parameterization by finite measure $\Lambda(dx) = x^2 \nu(dx)$ was used. The reader is referred to the recent

*Utrecht University, e-mail: A.V.Gnedin@uu.nl

†National T. Shevchenko University of Kiev, e-mail: iksan@unicyb.kiev.ua

‡National T. Shevchenko University of Kiev, e-mail: marynych@unicyb.kiev.ua

lecture notes [2, 4] for accessible introduction in the theory of lambda-coalescents and a survey.

After some number of collisions (merging events) Π_n enters the absorbing state with a sole cluster. Two basic characteristics of the speed of the coalescence are *the absorption time* τ_n and *the number of collisions* X_n . The large- n properties of τ_n and X_n are strongly determined by the concentration of measure ν on the unit interval near the endpoints of $[0, 1]$.

We suppose that ν has no mass at 1, which excludes forced termination of Π_∞ at independent exponential time. The coalescent is said to come down from infinity if $\Pi_\infty(t)$ has finitely many clusters for each $t > 0$ almost surely; then τ_n converge to a finite random variable τ_∞ which is the absorption time of Π_∞ . Otherwise, $\Pi_\infty(t)$ almost surely stays with infinitely many clusters for all t . There is a delicate criterion in terms of the rates $\lambda_{m,k}$ to distinguish between the two alternatives [26].

In this paper we shall study τ_n and X_n under the assumption that Π_∞ stays infinite due to infinitely many original clusters which do not engage in collisions before any given time $t > 0$. This family of lambda-coalescents can be characterized by the moment condition

$$\int_0^1 x \nu(dx) < \infty. \quad (2)$$

We call the collection of singleton clusters of $\Pi_\infty(t)$ *the dust component*. The dust component has a positive total frequency, meaning that the number of singletons within $\Pi_n(t)$ grows approximately linearly in n as $n \rightarrow \infty$.

The coalescents with dust component do not exhaust all coalescents which stay infinite. One distinguished example is the Bolthausen-Sznitman coalescent with $\nu(dx) = x^{-2}dx$ which stays infinite although (2) fails. Such examples on the border between ‘coming down from infinity’ and ‘possessing dust component’ are more of an exception if one considers e.g. measures ν satisfying a condition of regular variation near zero.

Under (2) every transition of Π_∞ will involve infinitely many singletons. This suggests that most of the collision events of Π_n will involve some of the original n clusters, for large n . Another way to express this idea is to say that in a tree representing the complete merging history of Π_n , most of the internal nodes are linked directly to one of n leaves. We will show that this intuition is indeed correct, to the extent that the behaviour of τ_n and X_n can be derived from that of analogous quantities associated with the evolution of the dust component. In turn, the total frequency of the dust component of Π_∞ undergoes a relatively simple process, which may be represented as $\exp(-S_t)$ where $S = (S_t)_{t \geq 0}$ is a subordinator. Similarly for Π_n , the engagement of original n clusters in their first collisions follows a Markovian process which has been studied in the context of regenerative composition structures derived from subordinators [13]. A coupling of Π_∞ with S will enable us to apply known results about the level-passage for subordinators, and about the asymptotics of regenerative composition structures.

The connection between Π_∞ and S was first explored in [11] in the special case when ν is a finite measure, hence subordinator S is a compound Poisson process. While in the present paper we are mainly interested in infinite ν , the case of finite ν is not excluded. Moreover, we will be able to extend the results of [11] by removing a condition on ν imposed in that paper.

In a recent paper by Haas and Miermont [18] results on counting collisions in the coalescent and counting blocks in the regenerative composition were derived separately in the context of absorption times of decreasing Markov chains. Our approach adds some insight to the connection between these two models, and it entails some delicate features like differentiating between collisions which involve some original clusters of Π_n and the collisions which do not.

The possible modes of behaviour of τ_n and X_n for large n are best illustrated by the family of coalescents driven by a beta measure

$$\nu(dx) = cx^{a-3}(1-x)^{b-1}dx, \quad a, b, c > 0. \quad (3)$$

These coalescents come down from infinity for $a < 1$ and stay infinite for $a \geq 1$. With the account of results of the present paper, we have the following list.

- (i) Case $0 < a < 1$. The limit law of $(X_n - (1-a)n)/n^{1/(2-a)}$ is $(2-a)$ -stable (see [16], and [21] for the case $b = 1$). The distribution of τ_∞ is unknown.
- (ii) Case $a = 1$. The instance $b = 1$ is the Bolthausen-Sznitman coalescent, for which the limit distribution of $\tau_n - \log \log n$ is standard Gumbel [9, 17], while X_n , suitably scaled and centered, converges weakly to a 1-stable distribution [7, 20]. The case $b \neq 1$ remains open.
- (iii) Case $1 < a < 2$. In the sequel we show that $(\tau_n - c \log n)/(\log n)^{1/2}$ (with suitable $c > 0$) converges weakly to a normal distribution, and that X_n/n^{2-a} converges to the exponential functional of a subordinator. The result about X_n was proved previously in [18], and in [21] in the case $b = 1$.
- (iv) Case $a \geq 2$. In the present paper we prove that normal limits hold for both τ_n and X_n with explicitly determined scaling and centering. In the case $a > 2$ these asymptotics were previously shown in [11]. In the case $a = 2$ the result for X_n was derived in [19].

2 The coalescent and singleton clusters

In the role of the state space of the coalescent Π_n with initially n clusters we take the set of partitions of $[n] := \{1, \dots, n\}$, in which every singleton cluster is classified as either *primary* or *secondary*. Under the *dust component* of $\Pi_n(t)$ we shall understand the collection of primary clusters. Every nonsingleton cluster of $\Pi_n(t)$ is regarded as secondary. For the notational convenience the clusters are written by increase of their minimal elements, the elements within the clusters are written in increasing order, and the secondary clusters are written in brackets. For instance, $1 \ (2) \ (3 \ 5 \ 6) \ 4 \ 7$, a partition of the set $[7]$, has three primary clusters and two secondary: $1, 4, 7$ and $(2), (3 \ 5 \ 6)$, respectively.

Introduce $\lambda_{m,1}$ as in (1) with $k = 1$. We have $\lambda_{m,1} < \infty$ by assumption (2).

We define the lambda-coalescent Π_n as a càdlàg Markov process with values in such partitions of $[n]$ and the initial state $1 \ 2 \ \dots \ n$ with n primary clusters. Each admissible transition is either merging some clusters in one cluster, or turning a primary singleton

cluster into secondary. From partition with m clusters, the transition rate for merging each particular k -tuple of m clusters in one is $\lambda_{m,k}$ ($2 \leq k \leq m$), and the transition rate for turning each particular primary singleton cluster into secondary singleton cluster is $\lambda_{m,1}$. For instance, the sequence of distinct states visited by Π_7 could be

$$1\ 2\ 3\ 4\ 5\ 6\ 7 \rightarrow 1\ 2\ (3\ 5\ 6)\ 4\ 7 \rightarrow 1\ (2)\ (3\ 5\ 6)\ 4\ 7 \rightarrow \\ 1\ (2\ 4)\ (3\ 5\ 6)\ 7 \rightarrow 1\ (2\ 3\ 4\ 5\ 6\ 7) \rightarrow (1\ 2\ 3\ 4\ 5\ 6\ 7).$$

Let $N_n(t)$ be the number of clusters in $\Pi_n(t)$. Then $N_n = (N_n(t))_{t \geq 0}$ is a nonincreasing Markov process, with the transition rate

$$\varphi_{m,k} := \binom{m}{k} \lambda_{m,k} \quad (4)$$

for jumping from m to $m - k + 1$, for $2 \leq k \leq m$. Turning a primary singleton cluster into a secondary singleton cluster does not cause a jump of N_n . The absorption time of Π_n can be recast as $\tau_n = \inf\{t : N_n(t) = 1\}$, and the number of collisions X_n is equal to the number of jumps the process N_n needs to proceed from n to 1 (which is 4 in the above example where the second transition does not alter the number of clusters).

Removing element n transforms partition of $[n]$ into partition of $[n - 1]$. For example, partitions $1\ (2\ 4)\ (3)$, $1\ (2)\ 3\ 4$ and $1\ (2)\ 3\ (4)$ all become $1\ (2)\ (3)$. Restricting in this way Π_n to $[n - 1]$, pointwise in $t \geq 0$, yields a stochastic copy of Π_{n-1} . This follows as in [23] since the rates satisfy the recursion $\lambda_{m,k} = \lambda_{m+1,k} + \lambda_{m+1,k+1}$ for all $1 \leq k \leq m$. Therefore we may define Π_n on the same probability space consistently for all n . Explicit realization will appear in the sequel.

The projective limit of the processes Π_n , $n \in \mathbb{N}$, is a Markov process Π_∞ starting at $t = 0$ with the infinite configuration of primary clusters $1\ 2\ \dots$, and assuming values in the space of partitions of the infinite set \mathbb{N} . Each partition $\Pi_\infty(t)$ has only primary singletons, namely those original clusters which do not engage in collisions up to time t . For a generic singleton, e.g. labelled 1, the time before its first collision has exponential distribution with parameter $\lambda_{1,1}$, and when such collision occurs infinitely many other clusters are engaged.

The differentiation of singletons of $\Pi_n(t)$ into primary and secondary becomes transparent by considering Π_n as restriction of Π_∞ on $[n]$. The secondary singletons of $\Pi_n(t)$ are the unique representatives in $[n]$ of some infinite clusters of $\Pi_\infty(t)$. The primary singletons of $\Pi_n(t)$ are also singletons in the partition $\Pi_\infty(t)$.

There is a construction of Π_∞ based on a planar Poisson point process in the strip $[0, 1] \times [0, \infty)$ with intensity measure $\nu(dx) \times dt$, see [2, 4, 23]. With each atom (t, x) one associates a transition of Π_∞ performed by tossing a coin with probability x for heads. To pass from $\Pi_\infty(t-)$ to $\Pi_\infty(t)$, the coin is tossed for each cluster of $\Pi_\infty(t-)$, then those clusters marked heads are merged in one, while the clusters marked tails remain unaltered. Although there are infinitely many transitions within any time interval if ν is an infinite measure, condition (2) ensures that Π_∞ does not terminate. In the case of finite ν transitions of Π_∞ occur at the epochs of Poisson process with rate $\nu([0, 1])$.

Let $N_n^*(t)$ be the number of primary clusters in $\Pi_n(t)$. By homogeneity properties of Π_n , the process $N_n^* = (N_n^*(t))_{t \geq 0}$ is a nonincreasing Markov process, jumping at rate $\varphi_{m,k}$

from m to $m - k$ for $1 \leq k \leq m$. Let

$$\tau_n^* := \inf\{t : N_n^*(t) = 0\}$$

be the random time when the last of n primary clusters disappears. For $1 \leq r \leq n$, let $K_{n,r}$ be the number of decrements of size r of (N_n^*) on the way from n to 0, let $K_n := \sum_{r=1}^n K_{n,r}$ be the total number of decrements of (N_n^*) , and let X_n^* be the number of non-unit decrements of (N_n^*) . Obviously,

$$X_n^* = K_n - K_{n,1}. \quad (5)$$

We call the clusters of partition $\Pi_n^*(\tau_n^*)$ that remain at time τ_n^* *residual*, and we denote R_n the number of residual clusters.

Processes N_n and N_n^* look very similar, thus at a first glance it might seem surprising that N_n^* is much easier to handle. The simplification comes from the identification of the sequence of decrements of N_n^* with the n th level of a regenerative composition structure [13], and further connection to the range of a subordinator. The main new contribution of the present paper is that N_n^* yields a good approximation for N_n for large n , thus X_n^* and τ_n^* are close to their counterparts X_n and τ_n . In one direction, the connection is quite obvious:

$$X_n^* \leq X_n, \quad N_n^*(t) \leq N_n(t), \quad \tau_n^* \leq \tau_n.$$

For instance, the first inequality holds since every collision taking at least two primary clusters contributes to X_n , and since with positive probability some $R_n \geq 2$ clusters remain at time τ_n^* when the last primary clusters disappears.

3 Coupling with a subordinator

Condition (2) implies that there exists a pure-jump subordinator $S = (S_t)_{t \geq 0}$ with the Laplace transform

$$\mathbb{E}(e^{-zS_t}) = e^{-t\Phi(z)}, \quad z \geq 0, \quad (6)$$

where the Laplace exponent is given by

$$\Phi(z) := \int_0^1 (1 - (1 - x)^z) \nu(dx).$$

The coalescent process will be represented in terms of passage of S through multiple exponentially distributed levels. We describe first the evolution of the dust component.

Let $\epsilon_1, \dots, \epsilon_n$ be independent of S i.i.d. standard exponential random variables, and let $\epsilon_{n:n} < \dots < \epsilon_{n:1}$ be their order statistics. It is not difficult to see that $\Phi(n)$ coincides with the probability rate at which the subordinator passes through the level $\epsilon_{n:n}$ from any state $S_t = s < \epsilon_{n:n}$. The following lemma extends this observation.

Lemma 3.1. *For $t \geq 0$, conditionally given $S_t = s$ with $s \in (\epsilon_{n:m+1}, \epsilon_{n:m})$ the subordinator is passing through $\epsilon_{n:m}$ at rate $\Phi(m)$, and is hitting at this passage each of the intervals $(\epsilon_{n:m-k+1}, \epsilon_{n:m-k})$ at rate $\varphi_{m,k}$, for $1 \leq k \leq m \leq n$.*

Proof. The proof exploits the Lévy-Khintchine formula (6) and the memoryless property of the exponential distribution. See computations around [13, Theorem 5.2] for details. \square

Now suppose that each of the primary clusters $1 \ 2 \dots n$ is given an exponential mark $\epsilon_1, \dots, \epsilon_n$, and that for every $t \geq 0$ the marks $\epsilon_j > S_t$ are associated with primary clusters j existing at time t . If t is a jump-time of S and the interval $(S_{t-}, S_t]$ covers exactly one mark ϵ_j , we interpret the event of passage through ϵ_j as turning the primary cluster j into secondary. If $(S_{t-}, S_t]$ covers at least two of the ϵ_j 's, we interpret this event as a collision which takes the corresponding primary clusters. Setting $N_n^*(t) := \#\{j \in [n] : \epsilon_j > S_t\}$ we obtain a process with desired rates $\varphi_{m,k}$ for transition from m to $m - k$, as it follows from the lemma. In particular, $\Phi(n) = \sum_{k=1}^n \varphi_{n,k}$ coincides with the total transition rate of the coalescent Π_n from the initial state $1 \ 2 \dots n$.

A *regenerative* ordered partition of the set $[n]$ is defined by sending i, j to the same block iff $T_{\epsilon_i} = T_{\epsilon_j}$, see [13]. The number of blocks of the partition is equal to the number of jumps of N_n^* prior to the absorption at state 0.

These evolutions of primary clusters are consistent in n . Assigning the exponential marks $\epsilon_1, \epsilon_2, \dots$ to infinitely many primary clusters $1 \ 2 \dots$ defines the initial state of the dust component. The frequency of the dust component of Π_∞ as time passes is the decaying process $(\exp(-S_t))_{t \geq 0}$.

One straightforward application of the representation by S concerns τ_n^* , the maximal lifetime of primary clusters in Π_n . Let

$$T_s := \inf\{t \geq 0 : S_t > s\}$$

be the first passage time through level $s \geq 0$. We can identify τ_n^* with $T_{\epsilon_{n:1}}$, hence connect the limit behaviour of τ_n^* to that of T_s for high levels s . Indeed, from the extreme-value theory it is known that $\epsilon_{n:1} - \log n$ converges in distribution, as $n \rightarrow \infty$, to a random variable with the Gumbel distribution. It is also known that the scaled and centered random variables $(T_s - g(s))/f(s)$ can converge in distribution only if the normalizing constant $f(s)$ goes to ∞ with s . Thus, $T_{\epsilon_{n:1}}$ and $T_{\log n}$ have the same limit law, if any. Moreover, it can be shown that $(T_s - g(s))/f(s)$ converges weakly to a given proper and nondegenerate probability law if and only if the same holds for $(T'_s - g(s))/f(s)$, where T'_s is the number of points within $[0, s]$ of a random walk which starts at 0 and has the generic step distributed like S_1 . See [5] (or Proposition 27 in [22]) for a complete list of limit distributions of T'_s and the conditions of convergence. Summarizing the above, we have

Proposition 3.2. *For constants $a_n > 0$ and $b_n \in \mathbb{R}$, if one of the random variables $(\tau_n^* - b_n)/a_n$ and $(T_{\log n} - b_n)/a_n$ converges weakly, as $n \rightarrow \infty$, to a nondegenerate proper distribution, then the other random variable converges weakly to this distribution too.*

To realize the full dynamics of Π_n in terms of the level-passage, a mark is assigned to each cluster according to the following rule. At time 0 the marks $\epsilon_1, \dots, \epsilon_n$ represent the primary clusters $1 \ 2 \dots n$. At time $t > 0$ there is some collection of marks on $[S_t, \infty)$ representing the clusters existing at this time. If at time $t > 0$ the subordinator passes through exactly k marks corresponding to some clusters $I_1, \dots, I_k \subset [n]$, then a new

cluster $I_1 \cup \dots \cup I_k$ is born and assigned a mark $S_t + \epsilon$, where ϵ is a copy of the unit exponential random variable, independent of S and all other marks assigned before t . For instance, if at the first passage time $t = T_{\epsilon_{n:n}}$ the subordinator jumps through exactly k levels $\epsilon_{j_1}, \dots, \epsilon_{j_k}$ out of $\epsilon_1, \dots, \epsilon_n$, then the secondary cluster $J = \{j_1, \dots, j_k\}$ is born (which is a singleton if $k = 1$) and assigned a mark exponentially distributed on $[S_t, \infty)$.

In particular, when S passes at some time t through only one mark, there is no change in $\Pi_n(t)$, and the mark of the corresponding singleton cluster is just re-assigned.

4 The absorption time

We wish to exploit the lifetime τ_n^* of primary clusters as approximation to the absorption time τ_n . At time τ_n^* the coalescent process is left with R_n residual clusters, whence the distributional identity

$$\tau_0 := 0, \quad \tau_n \stackrel{d}{=} \tau_n^* + \tilde{\tau}_{R_n}, \quad n \in \mathbb{N}, \quad (7)$$

where $\tilde{\tau}_m$ is assumed independent of (τ_n^*, R_n) and distributed like τ_m , for each $m \in \mathbb{N}_0$. To address the quality of approximation we need to estimate R_n .

We begin with some preparatory work. By the first transition the Markov chain N_n^* goes from n to a state with distribution $p_{n,k} := \varphi_{n,n-k}/\Phi(n)$, $0 \leq k \leq n-1$. Let $g_{n,k}$ be the probability that N_n^* ever visits state k , so in terms of the realization via subordinator, $g_{n,k} = \mathbb{P}(T_{\epsilon_{n:k+1}} < T_{\epsilon_{n:k}})$ is the probability that the interval $[\epsilon_{n:k+1}, \epsilon_{n:k}]$ intersects the range of S . An explicit formula for $g_{n,k}$ in terms of Φ is available (see [13], Equation (50)), but it is complicated and inconvenient for computations.

Lemma 4.1. *Suppose $(r_k)_{k \in \mathbb{N}}$ is a nonnegative sequence such that the sequence $\left(\frac{\Phi(k)r_k}{k}\right)_{k \in \mathbb{N}}$ is nonincreasing. Then the sequence $(a_n)_{n \in \mathbb{N}_0}$ defined by*

$$a_0 = 0, \quad a_n := \sum_{k=1}^n g_{n,k} r_k, \quad n \geq 1$$

satisfies the relation

$$a_n = O\left(\sum_{k=1}^n \frac{r_k \Phi(k)}{k}\right), \quad n \rightarrow \infty.$$

Proof. The assertion follows from Lemma 6.1 in the Appendix. Indeed, conditioning on the size of the first jump of N_n^* we see that the sequence (a_n) satisfies the recurrence

$$a_0 = 0, \quad a_n = r_n + \sum_{k=0}^{n-1} p_{n,k} a_k, \quad n \in \mathbb{N}.$$

To apply Lemma 6.1 we take $\psi_n = \Phi(n)$. Condition (C2) holds by the assumptions and condition (C1) follows from

$$\Phi(n) \sum_{k=0}^n (1 - k/n) p_{n,k} = \frac{1}{n} \sum_{k=0}^{n-1} (n - k) \varphi_{n,n-k} = \frac{1}{n} \sum_{k=1}^n k \varphi_{n,k} = \int_0^1 x \nu(dx) > 0. \quad (8)$$

□

Note that, since the function $s \mapsto \Phi(s)/s$ is nonincreasing, the sequence $\left(\frac{\Phi(k)r_k}{k}\right)$ is nonincreasing whenever (r_k) is itself nonincreasing.

Denote $\vec{\nu}(x) := \nu([x, 1])$, $x \in (0, 1)$.

Lemma 4.2. *If either of two equivalent conditions*

$$\int_0^1 x^{-1} dx \int_0^x \vec{\nu}(y) dy < \infty, \quad (9)$$

$$\sum_{k=1}^{\infty} \frac{\Phi(k)}{k^2} < \infty \quad (10)$$

holds then

$$\mathbb{E}R_n = O(1), \quad n \rightarrow \infty,$$

in which case the sequence of distributions of the R_n 's is tight.

Proof. The equivalence of (9) and (10) is established by repeated integration by parts.

In the genealogical history of each residual cluster there is the last secondary cluster appearing as a result of collision or switch involving some primary clusters. If secondary cluster b is born at some time $t \leq \tau_n^*$ of such an event, and if at this time some $j \geq 0$ other primary clusters co-exist, then b corresponds to a residual cluster provided that b and its followers do not collide with these j primary clusters or their followers before time τ_n^* . That is to say, b and the j primary clusters belong to distinct branches if the coalescent tree is cut at time τ_n^* . Let q_j be the probability that such cluster b corresponds to a residual cluster; restricting the coalescent to $j+1$ clusters it is seen that q_j indeed depends only on j . The consistency property of the coalescent with respect to the restrictions entails that q_j is decreasing in j . Averaging over the times when some primary clusters engaged we find the expected number of residual clusters

$$\mathbb{E}R_n = \sum_{j=0}^{n-1} g_{n,j} q_j. \quad (11)$$

Furthermore, given $S_t = s$, we have exactly j exponential marks of the primary clusters larger than s . The cluster b is assigned a new exponential mark $u = s + \epsilon$ which lies within each of the spacings in (s, ∞) generated by $\epsilon_{n:j}, \dots, \epsilon_{n:1}$ with the same probability $1/(j+1)$. If this spacing is $(\epsilon_{n:k+1}, \epsilon_{n:k})$ then b may correspond to a residual cluster only if (i) $T_{\epsilon_{n:k+1}} < T_u < T_{\epsilon_{n:k}}$ and (ii) b does not collide further with k primary clusters and their followers before time τ_n^* . If (i) occurs, condition (ii) is not sufficient for the correspondence since possible collisions with some of j primary clusters or their followers are ignored. This leads to the inequality

$$q_j \leq \frac{1}{j+1} \sum_{k=0}^j g_{j+1,k+1} p_{k+1,k} q_k, \quad 1 \leq j \leq n-1,$$

and $q_0 = 1$. Substituting $\varphi_{k,1} = k(\Phi(k) - \Phi(k-1))$ we obtain

$$\begin{aligned}
q_j &\leq \frac{1}{j+1} \sum_{k=1}^{j+1} g_{j+1,k} \frac{k(\Phi(k) - \Phi(k-1))}{\Phi(k)} q_{k-1} \\
&\leq \frac{c}{j+1} \sum_{k=1}^{j+1} (\Phi(k) - \Phi(k-1)) q_{k-1},
\end{aligned}$$

where Lemma 4.1 was applied with

$$r_k = \frac{k(\Phi(k) - \Phi(k-1)) q_{k-1}}{\Phi(k)}.$$

The required monotonicity condition holds since both q_k and $\Phi(k) - \Phi(k-1)$ are decreasing in k , the latter by concavity of Φ . Here and throughout c will denote a positive constant whose value is not important and may change from line to line.

Setting $a_j = (j+1)q_j$ and $b_j = c(\Phi(j+1) - \Phi(j))/(j+1)$, we obtain from the above

$$a_j \leq \sum_{k=0}^j b_k a_k, \quad j \in \mathbb{N}_0.$$

We want to show that the sequence (a_j) is bounded. To that end, let $M_j := \max_{i=0, \dots, j} a_i$, then also

$$M_j \leq \sum_{k=0}^j b_k M_k.$$

Since $\Phi(j)/j$ decreases we have $\Phi(j+1) - \Phi(j) \leq \Phi(j+1)/(j+1)$, which taken together with (10) implies that the series $\sum_{k=0}^{\infty} b_k$ converges, so we can choose

$$n_0 := \inf \{k \geq 0 : \sum_{i=k}^{\infty} b_i < 1/2\}.$$

If $\lim_{n \rightarrow \infty} M_n = \infty$ then

$$1 \leq \liminf_{n \rightarrow \infty} \frac{\sum_{k=0}^n b_k M_k}{M_n} = \liminf_{n \rightarrow \infty} \frac{\sum_{k=n_0}^n b_k M_k}{M_n} \leq \sum_{k=n_0}^{\infty} b_k \leq 1/2,$$

which is an obvious contradiction. Therefore (a_n) is bounded. From this

$$q_j \leq M_j/(j+1) \leq c/j.$$

Substituting this bound into (11) and applying Lemma 4.1 leads to the conclusion that $\mathbb{E}R_n$ remains bounded, as $n \rightarrow \infty$, by the virtue of (10). \square

Recall that the convergence of T_s in distribution always requires a scaling constant going to ∞ as $s \rightarrow \infty$. Under conditions of Lemma 4.2 the sequence of laws of τ_{R_n} is tight. Now from Proposition 3.2 and the decomposition (7) the following main result of this section emerges.

Theorem 4.3. *Suppose (9) holds. For some constants $a_n > 0$ and $b_n \in \mathbb{R}$, if one of the variables $(T_{\log n} - b_n)/a_n$ and $(\tau_n - b_n)/a_n$ converges weakly, as $n \rightarrow \infty$, to a nondegenerate proper distribution then the other variable converges weakly to the same distribution.*

The value of this result lies in the fact that the limit laws for T_s and the conditions of convergence are immediately translated into the convergence of τ_n . Normalizing and centering constants are known explicitly, see Proposition 27 in [22] or [5]. It follows that only stable laws and the Mittag-Leffler laws can appear as the limit distributions of τ_n .

If measure ν is finite the condition (9) obviously holds. In this case S is a compound Poisson process. Theorem 4.3 has been proved [11] under the assumptions that ν is not supported by a geometric sequence $(1 - x^k)_{k>0}$ (meaning that the law of S_1 is nonlattice) and that

$$\theta := \int_0^1 |\log x| \nu(dx) < \infty. \quad (12)$$

Theorem 4.3 shows that the result of [11] is still true without requiring (12).

Assumption (9) is not very restrictive since $\Phi(k) = o(k)$, $k \rightarrow \infty$, always holds. Concretely, suppose the right tail of ν has the property of regular variation at 0, that is

$$\vec{\nu}(x) \sim x^{-\gamma} \ell(1/x), \quad x \downarrow 0, \quad (13)$$

for some function ℓ of slow variation at ∞ , and $\gamma \in [0, 1]$. Then condition (9) is satisfied for $\gamma \in [0, 1)$. In the edge case $\gamma = 1$ the behaviour of ℓ is important, for instance (9) holds for $\ell(y) = (\log y)^{-\delta}$ if $\delta > 2$ and does not hold if $\delta \in (1, 2]$.

We use condition (9) to bound R_n , although we perceive that (9) can be omitted and the equivalence in Theorem 4.3 holds in full generality for the coalescents with dust component. Note that (9) is the local property of $\vec{\nu}$ near 0. More substantially, the limit law is affected by the decay at ∞ of the right tail of the distribution of S_1 , for which the behaviour of $\vec{\nu}$ near 1 is responsible. We illustrate this by two examples.

Example: normal limits. Assume in addition to (9) that

$$\mathbf{s}^2 := \text{Var}(S_1) = \int_0^1 |\log(1 - x)|^2 \nu(dx) < \infty.$$

Then as $n \rightarrow \infty$

$$\frac{\tau_n - \mathbf{m}^{-1} \log n}{(\mathbf{m}^{-3} \mathbf{s}^2 \log n)^{1/2}} \xrightarrow{d} \mathcal{N}(0, 1), \quad (14)$$

where $\mathbf{m} := \mathbb{E}S_1 = \int_0^1 |\log(1 - x)| \nu(dx)$.

This setting applies to beta coalescents mentioned in Introduction. We choose the constant in (3) to be $c = 1/B(a, b)$, where B is the beta function. The case $a > 2$ was settled in [11]. We focus on the previously open case $1 < a \leq 2$.

For $a = 2$ we compute the constants as

$$\mathbf{m} = b(b + 1)\zeta(2, b), \quad \mathbf{s}^2 = 2b(b + 1)\zeta(3, b),$$

where ζ is the Hurwitz zeta function.

For $a \in (1, 2)$ we have

$$\mathfrak{m} = \frac{a+b-1}{(a-1)(2-a)} \left(1 - (a+b-2) \{ \Psi(a+b-1) - \Psi(b) \} \right),$$

$$\mathfrak{s}^2 = \frac{a+b-1}{(a-1)(2-a)} \times \left(2 \{ \Psi(a+b-1) - \Psi(b) \} - (a+b-2) \{ (\Psi(a+b-1) - \Psi(b))^2 + \Psi'(b) - \Psi'(a+b-1) \} \right),$$

where Ψ is the logarithmic derivative of the gamma function. Finally, condition (9) holds since (13) is satisfied with $\gamma = 2 - a \in [0, 1)$ and constant function ℓ . Therefore, convergence (14) holds with the computed \mathfrak{m} and \mathfrak{s} .

Example: stable limits. Assume (9) and

$$\vec{\nu}(1 - e^{-y}) \sim y^{-\beta} L(y), \quad y \rightarrow \infty, \quad (15)$$

for some function L slowly varying at ∞ and $\beta \in (1, 2)$. Then

$$\frac{\tau_n - \mathfrak{m}^{-1} \log n}{\mathfrak{m}^{-(\beta+1)/\beta} c_{\lfloor \log n \rfloor}} \xrightarrow{d} \mathcal{S}(\beta), \quad n \rightarrow \infty, \quad (16)$$

where c_n is any sequence satisfying $\lim_{n \rightarrow \infty} nL(c_n)/c_n^\beta = 1$, and $\mathcal{S}(\beta)$ is the β -stable distribution with characteristic function

$$z \mapsto \exp\{-|z|^\beta \Gamma(1-\beta)(\cos(\pi\beta/2) + i \sin(\pi\beta/2) \operatorname{sgn}(z))\}, \quad z \in \mathbb{R}.$$

To illustrate, consider

$$\nu(dx) = \frac{x^{a-2} dx}{(1-x)|\log(1-x)|^d},$$

where $d \in (2, 3)$ and $a \in (d, d+1)$. Then (2) is satisfied, and condition (13) holds with $\gamma = d+1-a \in (0, 1)$ which implies (9). Condition (15) is fulfilled with $\beta = d-1 \in (1, 2)$. Therefore, the absorption time τ_n of such coalescent has limit law (16).

5 The number of collisions

5.1 Preliminaries

As an approximation to the number of collisions X_n we shall consider X_n^* , the number of jumps of N_n^* of size at least two. We will not be able to derive a complete result comparable with Theorems 5.3 or 4.3 because the universal criterion for convergence of X_n^* is not available. The cases when we know the behaviour of X_n^* (from [14, 15], [1] and [10]) are all covered by the assumption that ν satisfies (13). We shall also proceed in this

direction but exclude the case $\gamma = 1$ when $K_{n,1}$ is the term of dominating growth in the sum $K_n = \sum_{r=1}^n K_{n,r}$. By Karamata's Tauberian theorem [6] condition (13) with $\gamma < 1$ is equivalent to the analogous asymptotics of the Laplace exponent

$$\Phi(z) \sim \Gamma(1 - \gamma) z^\gamma \ell(z), \quad z \rightarrow \infty.$$

The case of finite ν appears when $\gamma = 0$ and Φ is an increasing bounded function.

The sequence (X_n) is nondecreasing and satisfies a distributional recurrence

$$X_1 = 0, \quad X_n \stackrel{d}{=} \tilde{X}_{n-J_n+1} + 1(J_n \geq 2), \quad n \geq 2, \quad (17)$$

where in the right-hand side J_n is independent of the \tilde{X}_i 's, $\tilde{X}_i \stackrel{d}{=} X_i$, and J_n is distributed like the first decrement of N_n^* , that is $\mathbb{P}(J_n = k) = p_{n,n-k}$ for $1 \leq k \leq n$. Similarly, the number X_n^* of collisions which involve at least two primary clusters satisfies

$$X_1^* = 0, \quad X_n^* \stackrel{d}{=} \tilde{X}_{n-J_n}^* + 1(J_n \geq 2), \quad n \geq 2, \quad (18)$$

with the convention $X_0^* = 0$. We may decompose X_n as

$$X_n = X_n^* + D_n \stackrel{(5)}{=} K_n - K_{n,1} + D_n \quad \text{a.s.} \quad (19)$$

where D_n is the number of collisions which take at most one primary cluster. Thus a collision contributes to D_n if either exactly one primary cluster merges with at least one secondary cluster, or at least two secondary and no primary clusters are merged.

Lemma 5.1. *We have*

$$\mathbb{E}D_n \leq c \sum_{k=1}^n (\Phi(k)/k)^2, \quad n \in \mathbb{N}. \quad (20)$$

In particular, if either of two equivalent conditions

$$\int_0^1 x^{-2} \left(\int_0^x \vec{\nu}(y) dy \right)^2 dx < \infty, \quad (21)$$

$$\sum_{k=1}^{\infty} (\Phi(k)/k)^2 < \infty \quad (22)$$

holds then the sequence of distributions of the D_n 's is tight.

Proof. The equivalence of (21) and (22) follows from [3, Proposition 1.4].

Choose some primary cluster b , to be definite let it be the cluster labelled 1, and suppose \tilde{X}_{n-1} is realised as the number of collisions among $n-1$ primary clusters $[n] \setminus \{b\}$ and their followers. Then $X_n = \tilde{X}_{n-1} + Z_n$, where Z_n is the indicator of the event that the first collision of b involves exactly one other cluster a . At the time of the merge of b with a the Markov chain N_n^* decrements by two or one, depending on whether a is primary or secondary. Let Y_n be the indicator of the event that the first involvement of b is either

turning b into secondary cluster, or a collision taking at most one other primary cluster and arbitrary number of secondary clusters. Clearly, $Y_n \geq Z_n$, therefore from (17)

$$X_n \stackrel{d}{\leq} \tilde{X}_{n-J_n} + Y_{n-J_n+1} + 1(J_n \geq 2), \quad (23)$$

where $\stackrel{d}{\leq}$ stands for ‘stochastically smaller’. Passing to expectations in (23), (18) and (19) we see that, for $d_n := \mathbb{E}D_n$, $y_n := \mathbb{E}Y_n$,

$$d_1 = 0, \quad d_n \leq \sum_{k=1}^{n-1} p_{n,k}(d_k + y_{k+1}), \quad n = 2, 3, \dots,$$

and iterating this inequality yields

$$d_1 = 0, \quad d_n \leq \sum_{j=1}^{n-1} g_{n,j} y_{j+1}, \quad n = 2, 3, \dots$$

By exchangeability, we have $y_n = (\mathbb{E}K_{n,1} + 2\mathbb{E}K_{n,2})/n$. Since

$$\mathbb{E}K_{n,1} = \sum_{k=1}^n g_{n,k} p_{k,k-1} = \sum_{k=1}^n g_{n,k} \frac{k(\Phi(k) - \Phi(k-1))}{\Phi(k)},$$

using Lemma 4.1 with $r_k = k(\Phi(k) - \Phi(k-1))/\Phi(k)$ yields

$$\mathbb{E}K_{n,1} \leq c \Phi(n), \quad n \in \mathbb{N}. \quad (24)$$

Using this, an inequality shown in Appendix and the monotonicity of Φ ,

$$\mathbb{E}K_{n,2} \stackrel{(30)}{\leq} c_1 \mathbb{E}K_{\lceil n/2 \rceil, 1} \stackrel{(24)}{\leq} c_2 \Phi(\lceil n/2 \rceil) \leq c_2 \Phi(n).$$

Thus

$$d_n \leq c \sum_{k=1}^n g_{n,k} \Phi(k)/k,$$

and using Lemma 4.1 with $r_k = c\Phi(k)/k$ results in (20). \square

5.2 The compound Poisson case

Assume that ν is a finite measure on $(0, 1)$, not supported by a geometric sequence of the form $(1 - x^k)_{k \geq 0}$, for some $x \in (0, 1)$. Since a linear time change of the coalescent does not affect the distribution of X_n we will not lose generality by assuming that ν is a probability measure on $(0, 1)$. Let $(W_k)_{k \in \mathbb{N}}$ be independent copies of a random variable W such that the law of $1 - W$ is ν . The subordinator S is then a unit rate compound Poisson process with the generic jump $|\log W|$ having some nonlattice law.

The variable $K_{n,r}$ introduced on p. 5 can be identified with the number of parts of size r in the regenerative composition of $[n]$ associated with S . Alternatively, $K_{n,r}$ has

interpretation in terms of the following occupancy model (see e.g. [10]). Consider a random discrete distribution

$$P_k := W_1 \dots W_{k-1}(1 - W_k), \quad k \in \mathbb{N},$$

with P_k thought of as a frequency of box k . Suppose n balls are thrown in infinitely many boxes, independently given (P_k) , with probability P_k of falling in box k for each ball. Then $K_{n,r}$ can be identified with the number of boxes occupied by exactly k out of n balls.

Introduce

$$\mathfrak{m} := \int_0^1 |\log(1 - x)| \nu(dx)$$

and for $1 \leq r \leq n$ let $\varkappa_{n,r} := \mathbb{E}K_{n,r}$.

Proposition 5.2.

- (a) If $\mathfrak{m} < \infty$ then for every $r = 1, 2, \dots$ the vector $(K_{n,1}, K_{n,2}, \dots, K_{n,r})$ converges weakly, as $n \rightarrow \infty$, to a proper multivariate distribution, and $\varkappa_{n,r} \rightarrow (\mathfrak{m}r)^{-1}$.
- (b) If $\mathfrak{m} = \infty$ then $\varkappa_{n,r} \rightarrow 0$, so $K_{n,r} \rightarrow 0$ in probability.

Proof. Part (a) was proved in [12], Theorem 3.3.

For (b) consider a random walk $(Q_j)_{j \geq 0}$ with $Q_0 = 0$ and the generic step $|\log W|$. Then $P_j = (1 - W_j) \exp(-Q_{j-1})$. Using $1 - x \leq e^{-x}$ with $x \in [0, 1]$, and substituting e^z for n , we reduce estimating $\varkappa_{n,1} = n \sum_{j \geq 1} P_j (1 - P_j)^{n-1}$ to estimating

$$\begin{aligned} \mathbb{E} \left(\sum_{j \geq 1} e^z P_j e^{-e^z P_j} \right) &= \mathbb{E} \left(\sum_{j \geq 1} (1 - W_j) \exp\{z - Q_{j-1} - e^{z - Q_{j-1}} (1 - W_j)\} \right) \\ &= \int_0^\infty f(z - y) dU(y), \end{aligned}$$

where $f(y) := \mathbb{E}\{(1 - W) \exp(y - e^y(1 - W))\}$ and $U(y) := \sum_{j \geq 0} \mathbb{P}\{Q_j \leq y\}$ is the renewal function of the random walk. The function f is nonnegative and integrable, since $\int_{-\infty}^\infty f(y) dy = 1$. Furthermore, the function $y \rightarrow e^{-y} f(y)$ is nonincreasing. It is known that these properties together ensure that f is directly Riemann integrable (see, for instance, the proof of Corollary 2.17 in [8]). When $\mathfrak{m} = \mathbb{E}|\log W_j| = \infty$, application of the key renewal theorem yields $\int_0^\infty f(z - y) U(dy) \rightarrow 0$, as $z \rightarrow \infty$, whence $\varkappa_{n,1} \rightarrow 0$.

For $r > 1$ the argument is similar, or one can use the estimate $\varkappa_{n,r} \leq c_r \varkappa_{n,1}$ shown in Appendix, Lemma 6.2. \square

The next theorem improves upon a result from [11] by removing condition (12).

Theorem 5.3. For constants $a_n > 0$ such that $\lim_{n \rightarrow \infty} a_n = \infty$, and $b_n \in \mathbb{R}$, whenever any of the variables

$$\frac{K_n - b_n}{a_n}, \quad \frac{X_n^* - b_n}{a_n} \quad \text{or} \quad \frac{X_n - b_n}{a_n}$$

converges weakly, as $n \rightarrow \infty$, to a nondegenerate proper distribution then all three variables converge weakly to this distribution.

Proof. Recall representation (19). Since ν is a probability measure we have $\Phi(k) < 1$, hence condition (22) is satisfied, and the sequence of laws of the D_n 's is tight by Lemma 5.1. By Proposition 5.2, the sequence of laws of the $K_{n,1}$'s is tight as well. By the assumption $a_n \rightarrow \infty$ the result follows. \square

From [10] it is known that, depending on the behaviour of $\vec{\nu}(x)$ near $x = 1$ there are five different modes of the weak convergence of, suitably normalized and centered, K_n . We do not exhibit all these cases here, rather provide an example borrowed from [10] to demonstrate a substantial role of the parameter $\theta = \int_0^1 |\log x| \nu(dx)$.

Example. Suppose ν has the right tail of the form

$$\vec{\nu}(x) = \frac{|\log x|^\rho}{1 + |\log x|^\rho}, \quad x \in (0, 1]$$

with $\rho > 0$. In the case $\rho \in (0, 1/2)$ we have $\theta = \infty$, and

$$\frac{X_n - \mathfrak{m}^{-1} \log n + (\mathfrak{m}(1 - \rho))^{-1} \log^{1-\rho} n}{c \log^{1/2} n} \xrightarrow{d} \mathcal{N}(0, 1), \quad n \rightarrow \infty,$$

where $\mathfrak{m} = \int_0^1 |\log(1 - x)| \nu(dx)$.

In the other case, when $\rho > 1/2$ (then $\theta < \infty$ for $\rho > 1$), the centering simplifies, so that

$$\frac{X_n - \mathfrak{m}^{-1} \log n}{c \log^{1/2} n} \xrightarrow{d} \mathcal{N}(0, 1), \quad n \rightarrow \infty.$$

Evolution of secondary particles. In the compound Poisson case the number V_t of secondary clusters of $\Pi_\infty(t)$ is finite, for each $t \geq 0$. The process $V = (V_t)_{t \geq 0}$ starts with $V_0 = 0$ and is a Markov chain with the transition rate $\varphi_{m,k} = \binom{m}{k} \lambda_{m,k}$ for jumping from m to $m - k + 1$, $0 \leq k \leq m$, $k \neq 1$. The rate for $k = 0$ is given by the same formula (1), and $\varphi_{m,0} < \infty$ because ν is finite. The $k = 0$ transition, resulting in increase of the number of secondary clusters by one, occurs when some (in fact, infinitely many) primary clusters merge without engagement of secondary clusters. The stationarity of V is a consequence of the existence of the dust component with infinitely many clusters.

It can be shown that the Markov chain V is positively recurrent and has a unique stationary distribution (π_m) found from the balance equation

$$\pi_m = \sum_{k=0}^{\infty} \pi_{m+k-1} \varphi_{m+k-1,k} \quad (25)$$

supplemented by the conditions $\pi_0 = 0$ and $\sum_{m=1}^{\infty} \pi_m = 1$.

Suppose for example that $\nu(dx) = dx$ is the Lebesgue measure on $[0, 1]$. In this case $\varphi_{m,k} = (m+1)^{-1}$. Equation (25) becomes $\pi_m = \sum_{j=m-1}^{\infty} \pi_j / (j+1)$. Differencing yields $\pi_m - \pi_{m+1} = \pi_{m-1}/m$, which is readily solved as

$$\pi_m = \frac{e^{-1}}{(m-1)!}, \quad m = 1, 2, \dots$$

so in this case the stationary distribution is shifted Poisson.

In contrast, the number of secondary clusters in the finite coalescent Π_n is not a Markov process, because the transition rates depend on the number of remaining primary particles.

5.3 The case of slow variation

Suppose (13) holds with $\gamma = 0$ and slowly varying $\ell(z) \rightarrow \infty$, $z \rightarrow \infty$. The Laplace exponent satisfies then $\Phi(z) \sim \ell(z)$. Suppose also that the subordinator has finite moments

$$\mathfrak{m} = \mathbb{E}S_1 = \int_0^1 |\log(1-x)|\nu(dx), \quad \mathfrak{s}^2 = \text{Var } S_1 = \int_0^1 |\log(1-x)|^2\nu(dx).$$

Choose the centering/scaling constants as

$$b_n = \frac{1}{\mathfrak{m}} \int_0^n \frac{\Phi(z)}{z} dz, \quad a_n = \sqrt{\frac{\mathfrak{s}^2}{\mathfrak{m}^3} \int_0^n \frac{\Phi^2(z)}{z} dz}.$$

In [1] it was shown that for $n \rightarrow \infty$

$$\mathbb{E}K_n \sim b_n, \quad \sqrt{\text{Var } K_n} \sim a_n,$$

and that the normal limit $(K_n - b_n)/a_n \xrightarrow{d} \mathcal{N}(0, 1)$ holds for various classes of functions ℓ . In particular, this includes functions of slow variation at infinity with asymptotics as diverse as

$$\ell(z) = \log(\log(\dots(\log(z))\dots)), \quad \ell(z) = \log^\beta z, \quad \ell(z) = \exp(\log^\beta z),$$

where $\beta > 0$.

The series (22) converges for arbitrary ℓ , hence by Lemma 5.1 $\mathbb{E}D_n = O(1)$. On the other hand, from (24) and by the properties of slowly varying functions [6]

$$\mathbb{E}K_{n,1} = O(\Phi(n)) = o(a_n).$$

It is immediate now that $(K_n - b_n)/a_n \xrightarrow{d} \mathcal{N}(0, 1)$ implies both $(X_n^* - b_n)/a_n \xrightarrow{d} \mathcal{N}(0, 1)$ and $(X_n - b_n)/a_n \xrightarrow{d} \mathcal{N}(0, 1)$.

Example: gamma subordinators. Consider the classical gamma subordinator with Laplace exponent $\Phi(z) = \alpha \log(1 + z/\beta)$, where $\alpha, \beta > 0$. The corresponding ν driving the coalescent has density

$$\nu(dx) = \frac{\alpha(1-x)^{\beta-1}}{|\log(1-x)|} dx.$$

The central limit theorem for K_n was proved by different methods in [14] and [1]. From this we conclude that the number of collisions also satisfies $(X_n - b_n)/a_n \xrightarrow{d} \mathcal{N}(0, 1)$, where the constants can be chosen as

$$a_n = \sqrt{\frac{\beta \log^3 n}{3}}, \quad b_n = \frac{\beta \log^2 n}{2}.$$

Example: beta(2, b)-coalescents. For this family $\nu(dx) = x^{-1}(1-x)^{b-1}dx$. The convergence of X_n to the standard normal distribution holds with scaling/centering constants

$$a_n = \sqrt{\frac{s^2}{3m^3} \log^3 n}, \quad b_n = \frac{\log^2 n}{2m},$$

where $m = \zeta(2, b)$, $s^2 = 2\zeta(3, b)$.

5.4 Regular variation with index $0 < \gamma < 1$.

A key distribution in this case is the law of the random variable

$$I = \int_0^\infty \exp(-\gamma S_t) dt,$$

known as the exponential functional of the subordinator γS . The distribution of I is uniquely determined by the moments

$$\mathbb{E}I^k = \frac{k!}{\prod_{i=1}^k \Phi(\gamma i)}.$$

From [15] (Theorem 4.1 and Corollary 5.2) $X_n^*/a_n \xrightarrow{d} I$, where $a_n = \Gamma(2-\gamma)n^\gamma \ell(n)$, and no centering is required. In fact, K_n/a_n and $K_{n,r}/a_n$ ($r \geq 1$) converge almost surely and in the mean.

To justify the convergence of X_n using (19) we need to estimate $\mathbb{E}D_n$. For $0 < \gamma < 1/2$ we have $\mathbb{E}D_n = O(1)$ since $\Phi(z) \sim c\ell(z)z^\gamma$, hence the series (22) converges. For $1/2 < \gamma < 1$ we have

$$\sum_{k=1}^n (\Phi(k)/k)^2 \sim c n^{2\gamma-1} \ell^2(n),$$

and for $\gamma = 1/2$ the latter sum, as a function of n , has the property of slow variation at infinity (see [6], Proposition 1.5.8). Thus in any case $D_n/a_n \rightarrow 0$ in probability. It follows that $X_n/a_n \xrightarrow{d} I$.

Example: beta(a, b)-coalescents with $1 < a < 2$. In this case

$$\frac{X_n}{n^{2-a}} \xrightarrow{d} \frac{\Gamma(a+b)}{(2-a)\Gamma(b)} \int_0^\infty \exp\{-(2-a)S_t\} dt, \quad n \rightarrow \infty.$$

This result was obtained in [18] by another method, and with a change of variables the equivalence with Theorem 7.1 from [21] in the case $b = 1$ can be established.

The subfamily of beta-coalescents with parameters $b = 2 - a$ was intensively studied. In the literature sometimes $\alpha := 2 - a$ is taken as parameter, so that ν in this notation becomes

$$\nu(dx) = x^{-\alpha-1}(1-x)^{\alpha-1}.$$

In this case N_n^* decrements like a random walk conditioned to hit 0 and, moreover, there is an explicit formula (see [13] p. 471)

$$g_{n,k} = \frac{(\alpha)_k (\alpha)_{n-k}}{(\alpha)_n} \binom{n}{k},$$

where $(\alpha)_k$ denotes the rising factorial. The variable K_n is then the number of blocks in Pitman's (α, α) -partition (or in the regenerative composition induced by excursions of a Bessel bridge [13]). We refer to [24] and [2] for further multiple connections of these beta-coalescents to various random processes.

6 Appendix

A linear recursion. For each $n \in \mathbb{N}$, let $(p_{n,k})_{0 \leq k \leq n}$ be a probability distribution with $p_{n,n} < 1$. Define a sequence $(a_n)_{n \in \mathbb{N}}$ as a (unique) solution to the recursion

$$a_n = r_n + \sum_{k=0}^n p_{n,k} a_k, \quad n \in \mathbb{N}, \quad (26)$$

with given $r_n \geq 0$ and the initial value $a_0 = a \geq 0$.

Lemma 6.1. *Suppose there exists a sequence $(\psi_n)_{n \in \mathbb{N}}$ such that*

$$(C1) \quad \liminf_{n \rightarrow \infty} \psi_n \sum_{k=0}^n (1 - k/n) p_{n,k} > 0,$$

$$(C2) \quad \text{the sequence } (\psi_k r_k / k)_{k \in \mathbb{N}} \text{ is non-increasing.}$$

Then (a_n) defined by (26) satisfies

$$a_n = O\left(\sum_{k=1}^n \frac{r_k \psi_k}{k}\right), \quad n \rightarrow \infty. \quad (27)$$

In particular, (a_n) is bounded if the series $\sum_{k=1}^{\infty} \frac{r_k \psi_k}{k}$ converges.

Proof. Write for simplicity p_k for $p_{n,k}$ and let $\pi_k = \sum_{j=0}^k p_j$. Using (C2) we have

$$\sum_{k=1}^n \frac{r_k \psi_k}{k} \pi_{k-1} \geq \frac{r_n \psi_n}{n} \sum_{k=1}^n \pi_{k-1} = r_n \psi_n \sum_{j=0}^{n-1} (1 - j/n) p_j.$$

By (C1) there exist $n_0 \in \mathbb{N}$ and $c > 0$ such that

$$c \sum_{k=1}^n \frac{r_k \psi_k}{k} \pi_{k-1} \geq r_n, \quad n \geq n_0. \quad (28)$$

From this, $x_n := c \sum_{k=1}^n r_k \psi_k / k$ satisfies

$$x_n \geq r_n + \sum_{k=1}^n x_k p_k, \quad n \geq n_0 \quad (29)$$

To check the latter, write

$$\begin{aligned}
r_n + \sum_{k=1}^n x_k p_k &= r_n + c \sum_{j=1}^n \sum_{k=j}^n \frac{r_j \psi_j}{j} p_k \\
&= r_n + c \sum_{j=1}^n \frac{r_j \psi_j}{j} (1 - \pi_{j-1}) \\
&= r_n + c \sum_{j=1}^n \frac{r_j \psi_j}{j} - c \sum_{j=1}^n \frac{r_j \psi_j}{j} \pi_{j-1} \\
&= x_n + r_n - c \sum_{j=1}^n \frac{r_j \psi_j}{j} \pi_{j-1} \stackrel{(28)}{\leq} x_n.
\end{aligned}$$

Set $x_0 := 0$. Subtracting (26) from (29) we see that $y_n := x_n + c_0 - a_n$ satisfies $y_n \geq \sum_{k=0}^n p_k y_k$ for $n \geq n_0$ and arbitrary c_0 . We can achieve that the recursion for y_n holds for all $n \in \mathbb{N}$ by choosing $c_0 \geq \max_{n \leq n_0} a_n$. But then it is easily shown by induction that $y_n \geq 0$ for all $n \in \mathbb{N}$, which implies the desired estimate of a_n . \square

Estimates for the occupancy counts. Let $(p_k)_{k \in \mathbb{N}}$ be a probability mass function. Consider the multinomial occupancy scheme in which n balls are thrown independently in boxes, with probability p_j for box $1, 2, \dots$. The expected number of boxes occupied by exactly r out of n balls is

$$\varkappa_{n,r} = \binom{n}{r} \sum_{j \geq 1} p_j^r (1 - p_j)^{n-r}, \quad 1 \leq r \leq n.$$

Lemma 6.2. *For fixed $r < s$ there exists a constant c such that*

$$\varkappa_{n,r} \geq c \varkappa_{2n,s}, \quad n \in \mathbb{N}. \tag{30}$$

Proof. Using $(1 - x)^{-1} \geq e^x$ for $x \in (0, 1)$,

$$\begin{aligned}
\frac{\binom{n}{r} x^r (1 - x)^{n-r}}{\binom{2n}{s} x^s (1 - x)^{2n-s}} &\geq c_1 \frac{s!}{2^s r!} (nx)^{r-s} (1 - x)^{s-r-n} \\
&\geq c_2 (nx)^{r-s} e^{nx/2} \\
&\geq c_2 \min_{y>0} y^{r-s} e^{y/2} \\
&= c_2 \left(\frac{e}{2(s-r)} \right)^{s-r}.
\end{aligned}$$

\square

The result extends immediately to the case of random (P_k) . This generalization was used in the proof of Proposition 5.2 with the P_j 's being the sizes of intervals obtained by splitting $[0, 1]$ at points of the range of the process $\exp(-S)$.

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